TENGSIZLIKLARNI TRIGONOMETRIK ALMASHTIRISHLAR YORDAMIDA ISBOTLASH.

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Annotatsiya. Ushbu tezisda tengsizliklarni trigonometrik almashtirishlar yordamida isbotlash usullari keltirilgan.

Kalit so'zlar. Tengsizlik, ayniyat.

Аннотация. В диссертации представлены методы доказательства неравенств с помощью тригонометрических подстановок.

Ключевые слова. Неравенство, тождество

Ba'zida tengsizlikni isbotlashda trigonometrik almashtirish olish yaxshi foyda beradi. Almashtirish qulay olinganda tengsizlik darhol isbotlanadigan, oddiy shaklga kelib qoladi. Shuningdek trigonometrik funksiyalarning yaxshi ma'lum bo'lgan xossalari yordam berishi mumkin.

Biz dastlab bunday almashtirishlarni kiritamiz, so'ng ma'lum bo'lgan trigonometrik ayniyatlar va tengsizliklarni kiritamiz, va nihoyat bir nechta olimpiada masalalarini muhokama qilamiz.

1 – teorema: Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo'lishi uchun quyidagi tenglikning bajarilishi zarur va yetarli

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1.$$

Isbot: Daslab shuni ta'kidlash joizki $\alpha=\beta=\gamma$ bo'lgan holda teoremaning tasdig'i o'rinlidir. Umumiylikka ziyon yetkazmasdan $\alpha\neq\beta$ deb faraz qilaylik. $0\prec\alpha+\beta\prec 2\pi$ bo'lganligi uchun $(-\pi,\pi)$ intervalda $\alpha+\beta+\gamma'=\pi$ shartni qanoatlantiruvchi γ' mavjud.

Qo'shish formulalari va $tgx=ctg(\frac{\pi}{2}-x)$ formulaga ko'ra

$$tg\frac{\gamma'}{2}=ctg\frac{\alpha+\beta}{2}=\frac{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}}{tg\frac{\alpha}{2}+tg\frac{\beta}{2}};$$

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma'}{2}+tg\frac{\gamma'}{2}tg\frac{\alpha}{2}=1 \qquad (1)$$

Tenglik o'rinli bo'ladi. Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1 \qquad (2)$$

tenglik o'rinli bo'lsin.

Biz isbotlaymizki $\gamma'=\gamma$ va bu bizga α,β,γ lar biror uchburchak burchaklari ekanligini beradi. (1) dan (2) ni ayirib $tg\frac{\gamma}{2}=tg\frac{\gamma'}{2}$ ni hosil qilamiz. Shuning uchun $\left|\frac{\gamma-\gamma'}{2}\right|=k\pi, k\geq 0, k\in Z$. Ammo $\left|\frac{\gamma-\gamma'}{2}\right|\leq\left|\frac{\gamma}{2}\right|+\left|\frac{\gamma'}{2}\right|\prec\pi$ tengsizlik o'rinli. Demak, $k=0$, shuning uchun $\gamma'=\gamma$. Tasdiq isbotlandi.

2 – teorema: Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo'lishi uchun quyidagi tenglikning bajarilishi zarur va yetarli

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1$$

Isbot: $0\prec\alpha+\beta\prec 2\pi$ bo'lganligi uchun shunday $\gamma'\in(-\pi,\pi)$ mavjudki $\alpha+\beta+\gamma'=\pi$ tenglik o'rinli bo'ladi. Ko'paytmani yig'indiga keltirish va ikkilangan burchak formulalariga asosan quyidagi munosabatlar o'rinli

$$\sin^2\frac{\gamma'}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma'}{2}=\cos\frac{\alpha+\beta}{2}(\cos\frac{\alpha+\beta}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2})=$$

$$=\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=\frac{(1-2\sin^2\frac{\alpha}{2})+(1-2\sin^2\frac{\beta}{2})}{2}=1-\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}).$$

Shunday qilib,

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma'}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1 \qquad (3)$$

Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma'}{2}=1 \qquad (4)$$

tenglik o'rinli bo'lsin. (3) dan (4) ni ayirib,

$$\sin^2\frac{\gamma}{2}-\sin^2\frac{\gamma'}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}(\sin\frac{\gamma}{2}-\sin\frac{\gamma'}{2})=0 \text{ ya'ni}$$

$$(\sin\frac{\gamma}{2}-\sin\frac{\gamma'}{2})(\sin\frac{\gamma}{2}+\sin\frac{\gamma'}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2})=0$$

munosabat hosil qilamiz.

$$\sin\frac{\gamma}{2}+\sin\frac{\gamma'}{2}+\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta'}{2}=\sin\frac{\gamma}{2}+\cos\frac{\alpha-\beta}{2}$$

Ravshanki bu ifoda musbat qiymatlar qabul qiladi. Shuning uchun

$\sin\frac{\gamma}{2}=\sin\frac{\gamma'}{2}$, ya'ni $\gamma=\gamma'$ bo'ladi. Demak $\alpha+\beta+\gamma=\pi$. Tasdiq isbotlandi.

Almashtirishlar

1. Faraz qilaylik α,β,γ lar uchburchakning ichki burchaklari bo'lsin. Quyidagicha almashtirishni qaraylik

$$A=\frac{\pi-\alpha}{2};\ B=\frac{\pi-\beta}{2};\ C=\frac{\pi-\gamma}{2}.$$

Ravshanki $A+B+C=\pi$ va $0\le A,B,C\prec\frac{\pi}{2}$. Bu almashtirish bizga biror masalani hal qilishda istalgan uchburchak o'rniga o'tkir burchakli uchburchakni qarash imkonini beradi. Quyidagi munosabat o'rinli ekanligini ta'kidlash joiz:

$$\sin\frac{\alpha}{2}=\cos A\ ;\ \cos\frac{\alpha}{2}=\sin A\ ;\ tg\frac{\alpha}{2}=ctgA\ ;\ ctg\frac{\alpha}{2}=tgA$$

2. Faraz qilaylik x,y,z lar musbat haqiqiy sonlar bo'lsin. U holda tomonlari uzunliklari $a=x+y$; $b=y+z$; $c=x+z$ lardan iborat bo'lgab uchburchak mavjud. $S=x+y+z$ bo'lsa, $(x,y,z)=(S-a,S-b,S-c)$. Shartga ko'ra x,y,z lar musbatligi uchun $S-a,S-b,S-c$ lar uchburchak tengsizligini qanoatlantiradi.

3. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ac=1$ shartni qanoatlantirsin. Biz ushbu $f:(0;\frac{\pi}{2})\to(0;+\infty)$, $f(x)=tgx$ funksiya yordamida quyidagi almashtirish kiritishimiz mumkin

$$a=tg\frac{\alpha}{2};\ b=tg\frac{\beta}{2};\ c=tg\frac{\gamma}{2}\ ;$$

bunda α,β,γ lar biror uchburchakning burchaklari.

4. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ac=1$ shartni qanoatlantirsin.

1 va 3 larga ko'ra quyidagilarga egamiz

$$a = ctgA\ ;\ b = ctgB;\ c = ctgC\ ;$$

bunda A, B, C o'kir burchakli uchburchakning burchaklari.

5. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ac=abc$ shartni qanoatlantirsin. Bu tenglikning ikkala tarafini a,b,c sonlarning ko'paytmasiga bo'lib, quyidagiga ega bo'lamiz $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=1$. 3 ga ko'ra quyidagicha almashtirish olamiz

$$\frac{1}{a} = tg\frac{\alpha}{2};\ \frac{1}{b} = tg\frac{\beta}{2};\ \frac{1}{c} = tg\frac{\gamma}{2};$$

ya'ni

$$\frac{1}{a} = ctg\frac{\alpha}{2};\ \frac{1}{b} = ctg\frac{\beta}{2};\ \frac{1}{c} = ctg\frac{\gamma}{2}$$

bunda α, β, γ lar biror uchburchakning burchaklari.

Endi trigonometrik almashtirishlarning tadbiqlariga misollar keltiramiz.

1 – misol: Faraz qilaylik musbat x, y, z sonlar $x+y+z=xyz$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang.

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \le \frac{3}{2}$$

Bu masalada talabaning xayoliga eng birinchi $f(t)=\frac{1}{\sqrt{1+t^2}}$ funksiya uchun Yensen tengsizligini qo'llash kelishi mumkin. Ammo bu f funksiya R^+ to'plamda yuqorida qavariq emas. Ammo shunisi qiziqarliki $f(tg\theta)$ funksiya yuqoriga qavariq

Isbot: Quyidagicha almashtirish olaylik

$$x = tgA;\ y = tgB;\ z = tgC;\ A, B, C \in (0;\frac{\pi}{2}).$$

Ushbu $1+tg^2\alpha=\frac{1}{\cos^2\alpha}$ $\cos\alpha \ne 0$ ayniyatga ko'ra berilgan tengsizlik quyidagicha ko'rinishni oladi

$$\cos A + \cos B + \cos C \le \frac{3}{2}$$

Quyidagi $tg(\pi - C) = -z - \frac{x+y}{1-xy} = tg(A+B)$ va $\pi - C,$ $(A+B) \in (0;\pi)$ munosabatlardan $\pi - C = A+B$ yoki $A+B+C=\pi$ tenglikni olamiz. Demak istalgan ABC uchburchak uchun $\cos A + \cos B + \cos C \le \frac{3}{2}$ tengsizlik isbot qilsak yetarli ekan. Bu esa quyidagi munosabatdan kelib chiqadi.

$$3 - 2(\cos A + \cos B + \cos C) = (\sin A - \sin B)^2 + (\cos A + \cos B - 1)^2 \ge 0 .$$

2 – misol: Faraz qilaylik musbat x, y, z sonlar $x+y+z=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang.

$$\sqrt{\frac{xy}{z+xy}} + \sqrt{\frac{yz}{x+yz}} + \sqrt{\frac{zx}{y+zx}} \le \frac{3}{2}$$

Isbot: Yuqoridagi tengsizlik ushbu tengsizlikka teng kuchli

$$\sqrt{\frac{zx}{(y+z)(y+x)}} + \sqrt{\frac{yz}{(x+y)(x+z)}} + \sqrt{\frac{xy}{(z+x)(z+y)}} \le \frac{3}{2}$$

Bu tengsizlikning uchta hadini $\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}, \sin\frac{\gamma}{2}$ larga almashtiramiz va demak, ushbu $\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2}$ tengsizlikni isbotlashimiz kerak. Bu tengsizlikning o’rinli ekanligi ravshan. (Yensen tengsizligidan osongina kelib chiqadi).

Tengsizliklarni trigonometrik almashtirishlar yordamida isbotlash talabalarda olimpiada masalalarni funksiyaning xossalari yordamida yechishga asosiy matematik bilim, ko’nikma va malakalarni shakllantirishning asosiy shakli sifatida qaraladi. Misollarni yechishda matematik bilimlarning o’rni juda katta ayniqsa trigonometrik shakl almashtirishlar undagi belgilashlar va funksiyaning xossalari yordamida yechiladigan masalalar talabalarning fikrlash doirasini kengaytirish, matematik qobiliyatlarini o’stirish, fanga bo’lgan qiziqishlarini oshirish mumkin.

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