H_{11} OPERATORNING XOS QIYMATLARI

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Annotatsiya. Mazkur maqolani tahlil qilish jarayonida operatorning Fredholm determinanti operatorlarning muhim spektrdan tashqaridagi xos qiymatlarini va ularning karraliklarini *zarur va yetarli*ligini o'rganamiz. H_{11} operatorning xos qiymatlari va ularning karraliklari o'rganildi.

Kali so'zi. operatorning karrali xos qiymati, zarur va yetarli, Fredholm determinanti, Adamar tengsizli, operatorning yadrosi Veyl teoremasi

Biz $L_2[-\pi,\pi]$ Hilbert fazosida

$$H_{00} f_0 = af_0, \quad H_{01} f_1 = \int b(y) f_1(y)dy, \quad (H_{10} f_0)(x) = b(x) f_0,$$

$$(H_{11} f_1)(x) = u(x) f_1(x) + \int_{-\pi}^{\pi} K(x,y) f_1(y)dy. \qquad (1.1)$$

(1.1) formula bilan aniqlanuvchi o'z-o'ziga qo'shma chegaralangan $H = H_{11}$ operatorni, ya'ni

$$(Hf)(x) = u(x) f(x) + \int_{-\pi}^{\pi} K(x,y) f(y)dy \qquad (1.2)$$

ni qaraymiz va uning xos qiymatlarini o'rganamiz.

Muhim spektr haqidagi Veyl teoremasidan kelib chiqadiki H operatorning muhim spektri $\sigma_{ess}(H)$ yadro $K(x,y)$ ga bog'liq emas va u funksiyaning qiymatlar to'plamidan iborat, ya'ni $\sigma_{ess}(H) = Imu.$

Biz $\Delta(\lambda, z), z \in C \setminus \sigma_{ess}(H), \lambda \in R$ bilan $I + \lambda K(z),$ operatorning Fredholm determinantini belgilaymiz.

Bu yerda $I-$ birlik operator, $K(z)-$ esa $L_2[-\pi,\pi]$ Hilbert fazosidagi integral operator bo'lib, uning yadrosi quyidagicha aniqlanadi:

$$K(x,y;z) = K(x,y)(u(x)-z)^{-1/2}(u(y)-z)^{-1/2}. \qquad (1.3)$$

1.1-teorema. *Biror* $z_0 \in C \setminus \sigma_{ess}(H)$ *soni* H *operatorning* n *karrali xos qiymati bo'lishi uchun u* $\Delta(1,z)$ *funksiyaning* n *karrali noli bo'lishi zarur va yetarli.*

$I + \lambda K(z)$ operatorning Fredholm determinantini $\Delta(\lambda, z)$ va Fredholm minori $D(x, y; \lambda, z)$ quyidagicha qator ko'rinishida tasvirlanadi.

$$\Delta(\lambda, z) = 1 + \sum_{n=1} \frac{\lambda^n}{n!} B_n(z), \qquad (1.4)$$

$$B_n(z) = \int \begin{vmatrix} K(x_1, x_1) & K(x_1, x_2) & \ldots K(x_1, x_n) \\ K(x_2, x_1) & K(x_2, x_2) & \ldots K(x_2, x_n) \\ \vdots & \vdots & \vdots \\ K(x_n, x_1) & K(x_n, x_2) & \ldots K(x_n, x_n) \end{vmatrix} \prod_{i=1}^{n} \frac{dx_i}{u(x_i) - z}. \qquad (1.5)$$

$$D(x, y; \lambda, z) = \frac{\lambda K(x, y)}{u(x) - z} + \sum_{n=1}^{\infty} \frac{\lambda^{n+1} B_n(x, y; z)}{n!}, \qquad (1.6)$$

$$B_n(x, y; z) = \frac{1}{u(x) - z} \int \begin{vmatrix} K(x, y) & K(x, x_1) & \ldots K(x, x_n) \\ K(x_1, y) & K(x_1, x_1) & \ldots K(x_1, x_n) \\ \vdots & \vdots & \vdots \\ K(x_n, y) & K(x_n, x_1) & \ldots K(x_n, x_n) \end{vmatrix} \prod_{i=1}^{n} \frac{dx_i}{u(x_i) - z}. \qquad (1.7)$$

Hisoblashlarni soddalashtirish maqsadida biz quyidagi belgilashlarni kiritamiz:

$$B_n(z) = \int Q_n(x_1, x_2, \ldots, x_n; z) dx_1 \ldots dx_n,$$

$$Q_n(x_1, x_2, \ldots, x_n; z) = \begin{vmatrix} K(x_1, x_1) & K(x_1, x_2) & \ldots K(x_1, x_n) \\ K(x_2, x_1) & K(x_2, x_2) & \ldots K(x_2, x_n) \\ \vdots & \vdots & \vdots \\ K(x_n, x_1) & K(x_n, x_2) & \ldots K(x_n, x_n) \end{vmatrix} \prod_{i=1}^{n} \frac{1}{u(x_i) - z}.$$

Bu yerda indeks n determinantning tartibini bildiradi. Bundan tashqari har bir $s, p \in N$ natural sonlar uchun quyidagi belgilashlarni kiritamiz:

$$B_s(x_1, \ldots, x_p; z) = \int Q_{s+p}(x_1, \ldots, x_{s+p}; z) dx_{p+1} \ldots dx_{p+s}.$$

Xususan $p=0$ bo'lsa, $B_s(x_1,\ldots,x_p;z)$ bilan $B_s(z)$ ustma-ust tushadi. Bu holda $\Delta(\lambda,z)$ ning $p-$ tartibli minori quyidagi ko'rinishga ega bo'ladi:

$$D\begin{pmatrix} x_1,\ldots,x_p;z \\ x_1,\ldots,x_p;\lambda \end{pmatrix} = D(x_1,\ldots,x_p;\lambda,z) = \sum_{s=0}^{\infty}\frac{\lambda^{p+s}}{s!}B_s(x_1,\ldots,x_p;z). \qquad (1.8)$$

Diterminantlar uchun Adamar tengsizligi yordamida ko'rsatish mumkinki (1.4), (1.6) va (1.8) qatorlar absolyut va tekis yaqinlashadi, agarda $z\in G,\lambda\in\cup_r=\{\lambda\in C;|\lambda|\le r\}$ bo'lsa. Bu yerda G soha $C\setminus\sigma_{ess}(H)$ dagi ixtiyoriy kompakt to'plam, $r>0-$ ixtiyoriy son.

H operatorning xos qiymatlari bilan Fredholm determinanti $\Delta(1,z)$ ning nollari orasida uzviy bog'lanish mavjud, ya'ni quyidagi tasdiq o'rinli.

1.1-lemma. *Faraz qilaylik $\mu=-1$ soni $K(z)$ operatorning n karrali xos qiymati bo'lsin. U holda quyidagi munosabatlar o'rinli:*

a) $k=0,1,\ldots,n-1$ *lar uchun* $\frac{\partial^k\Delta(\lambda,z)}{\partial\lambda^k}|_{\lambda=1}=0$,

b) $\frac{\partial^n\Delta(\lambda,z)}{\partial\lambda^n}|_{\lambda=1}=\Delta_\lambda^{(n)}(1,z)\neq 0.$

c) *barcha* $x_1,\ldots,x_n\in[-\pi;\pi]$ *lar uchun* $[\Delta_\lambda^{(n)}(1,z)]^{-1}D(x_1,\ldots,x_n;z,1)\ge 0$ *tengsizlik o'rinli.*

1.2-lemma. *Barcha* $p\in N$ *natural sonlar uchun quyidagi tengliklar o'rinli:*

$$\frac{\partial D(x_1,\ldots,x_p;1,z)}{\partial z} =$$

$$\sum_{j=1}^{p}\frac{D(x_1,\ldots,x_p;1,z)}{u(x_j)-z}+\int\frac{D(x_1,\ldots,x_{p+1};1,z)dx_{p+1}}{u(x_{p+1})-z}. \qquad (1.9)$$

(1.9) tenglik $p=0$ quyidagi ko'rinishni oladi.

$$\frac{d\Delta(1,z)}{dz}=\int\frac{D(x;1,z)}{u(x)-z}dx. \qquad (1.10)$$

1.2-lemmaning isboti. Dastlab (1.10) tenglikni isbotlaymiz. $s-$ tartibli determinantlarning elementar xossalaridan hamda (1.3) tenglikdan $\det\{K(x_i,x_j;z)\}_s$ uchun quyidagiga ega bo'lamiz:

$$\det\{K(x_i, x_j; z)\}_s = \det\{K(x_i, x_j)\}_s \prod_{j=1}^{s}(u(x_j) - z)^{-1}. \qquad (1.11)$$

Funksiya $\det\{K(x_i, x_j; z)\}_s = Q_s(x_1, ..., x_s; z)$ o'zining argumentlari $x_1, \ldots, x_s \in [-\pi; \pi]$ larga nisbatan simmetrikdir. Undan z bo'yicha xususiy hosila hisoblaymiz:

$$\frac{\partial Q_s(x_1, \ldots, x_s; z)}{\partial z} = \sum_{k=1}^{s} \frac{Q_s(x_1, \ldots, x_s; z)}{u(x_k) - z}. \qquad (1.12)$$

$B_s(z)$ ning aniqlanishiga ko'ra va (1.12) formuladan quyidagiga ega bo'lamiz:

$$B'_s(z) = \sum_{k=1}^{s} \int Q_s(x_1, \ldots, x_s; z)(u(x_k) - z)^{-1} dx_1 \ldots dx_s.$$

$Q_s(x_1, \ldots, x_s; z)$ funksiya o'zining argumentlari $x_1, \ldots, x_s \in [-\pi; \pi]$ larga nisbatan simmetrikdir. Bu integralda $x_k = x_1$ almashtirish kiritib quyidagini hosil qilamiz:

$$B'_s(z) = s\int Q_s(x_1, \ldots, x_s; z)(u(x_1) - z)^{-1} dx_1 \ldots dx_s = \qquad (1.13)$$

$$= s\int [\int Q_s(x_1, \ldots, x_s; z) dx_2 \ldots dx_s] \frac{dx_1}{u(x_1) - z} = s\int \frac{B_{s-1}(x_1; z) dx_1}{u(x_1) - z}.$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} B_n(z)$$

qator istalgan $G \subset C \setminus \sigma_{ess}(H)$ kompaktda tekis yaqinlashganligi uchun uni z bo'yicha hadlab differensiallash mumkin:

$$\frac{d\Delta(1, z)}{dz} = \sum_{s=1}^{\infty} \frac{1}{s!} B'_s(z).$$

(1.13) dan foydalangan holda yig'indi va integrallashning o'rnini almashtirib quyidagini olamiz:

$$\frac{d\Delta(1, z)}{dz} = \int (\sum_{s=0}^{\infty} \frac{B_s(x; z)}{s!}) \frac{dx}{u(x) - z} = \int \frac{D(x; 1, z)}{u(x) - z} dx.$$

Shunday qilib, (1.10) tenglik isbot bo'ldi.

(1.9) tenglikni isbotlash uchun (1.6) qatorni hadlab differensiallaymiz

$$D'_z(x_1, \ldots, x_p; 1, z) = \sum_{j=1}^{p} \frac{1}{u(x_j) - z} \sum_{s=0}^{\infty} \frac{B_s(x_1, \ldots, x_p; z)}{s!} +$$

$$+\sum_{s=1}^{\infty}\frac{1}{s!}\sum_{k=p+1}^{p+s}\int\frac{Q_{p+s}(x_1,\dots,x_{p+s};z)}{u(x_k)-z}dx_{p+1}...dx_{p+s}.$$

$Q_{p+s}(x_1,\dots,x_{p+s};z)$ funksiyaning simmetrikligidan foydalangan holda integralda $x_k = x_{p+1}$ almashtirish kiritib quyidagiga ega bo'lamiz:

$$D'_z(x_1,\dots,x_p;1,z)=\sum_{j=1}^{p}\frac{D(x_1,\dots,x_p;1,z)}{u(x_j)-z}+$$

$$+\sum_{s=1}^{\infty}\frac{s}{s!}\int[\int Q_{p+s}(x_1,\dots,x_{p+s};z)dx_{p+2}\dots dx_{p+s}]\frac{dx_{p+1}}{u(x_{p+1})-z}.$$

So'nggi qo'shiluvchida yig'indi va integralning o'rinlarini almashtirib (1.9) tenglikning isbotiga ega bo'lamiz. 1.2-lemma to'la isbotlandi.

1.1-teoremaning isboti. H operatorning o'z-o'ziga qo'shma ekanligidan uning xos qiymatlari, jumladan $\Delta(1,z)$ funksiyaning nollari haqiqiy ekanligi kelib chiqadi. Shuning uchun faqat haqiqiy z larni qarash yetarli. Tenglamalar

$$(Hf)(x)=z_0 f(x) \quad \text{va} \quad (K(z_0)g)(x)=-g(x)$$

bir xil sondagi chiziqli bog'lanmagan yechimlarga ega, shuning uchun quyidagi tasdiq o'rinli:

1.3-lemma. *Biror* $z_0\in R\setminus\sigma_{ess}(H)$ *soni* H *operatorning xos qiymati bo'lishi uchun* $\mu=-1$ *soni* $K(z_0)$ *operatorning xos qiymati bo'lishi zarur va yetarli. Bundan tashqari* z_0 *xos qiymatning karraligi bilan* $\mu=-1$ *xos qiymatning karraligi ustma-ust tushadi.*

1.4-lemma. *Biror* $z_0\in R\setminus\sigma_{ess}(H)$ *soni* $\Delta(1,z)$ *funksiyaning* n *karrali noli bo'lishi uchun* $\mu=-1$ *soni* $K(z_0)$ *operatorning* n *karrali xos qiymati bo'lishi zarur va yetarli.*

1.4-lemmaning isboti. Yetarliligi. Faraz qilaylik $\mu=-1$ soni $K(z_0)$ operatorning n karrali xos qiymati bo'lsin. U holda 1.3 va 1.1-lemmalardan $\Delta(1,z_0)=0$ ekanligi kelib chiqadi. (1.6), (1.9) va (1.10) lardan barcha $k=\overline{1,n-1}$ larda $\Delta_z^{(k)}(1,z_0)=0$ ekanligi kelib chiqadi. Agar $k=n$ bo'lsa biz quyidagiga ega bo'lamiz:

$$\frac{d^n\Delta(z;1)}{d^n z}|_{z=z_0} = \int D(x_1,\ldots,x_n;1,z_0)\prod_{k=1}^{n}\frac{dx_k}{u(x_k)-z_0} =$$

$$= \Delta_\lambda^{(n)}(1,z_0)\int |\det\{\varphi_j(x_i)\}_n|^2 \prod_{k=1}^{n}\frac{dx_k}{u(x_k)-z_0}. \qquad (1.14)$$

Bu yerda oxirgi tenglik (1.7) tenglikdan kelib chiadi. z_0 soni u funksiyaning qiymatlar sohasiga tushmaganligi uchun barcha $x\in[-\pi;\pi]$ larda $u(x)-z_0$ ishorasini o'zgartirmaydi. Shuning uchun (1.15) da qatnashuvchi integral noldan farqli bo'ladi. 1.1-lemmaga ko'ra $\Delta_\lambda^{(n)}(1,z_0)\neq 0$ bo'ladi. Shuning uchun hamda (1.15) dan kelib chiqadiki $z=z_0$ soni $\Delta(1,z)$ funksiyaning $n-$ karrali noli bo'ladi.

Zaruriyligi. Faraz qilaylik, $z_0\in R\setminus\sigma_{ess}(H)$ soni $\Delta(1,z)$ funksiyaning $n-$ karrali noli bo'lsin. U holda 1.1 va 1.3-lemmalardan kelib chiqadiki $\mu=-1$ soni $K(z_0)$ operatorning xos qiymati bo'ladi. Faraz qilaylik, r soni $\mu=-1$ xos qiymatning karraligi bo'lsin. U holda yuqorida isbotlanganiga ko'ra $r=n.$ 1.4-lemma isbot bo'ldi.

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